

Review of Electrostatics (Cont'd)

Separation of Variables

This is a method of solving the Laplace equation (as well as other homogeneous linear partial differential equations). It depends on the separability of Laplacian along the coordinate directions of a coordinate system.

"Cartesian Coordinates"

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(x, y, z) = X(x)Y(y)Z(z) \Rightarrow YZ \frac{d^2 X}{dx^2} + ZX \frac{d^2 Y}{dy^2} + XY \frac{d^2 Z}{dz^2} = 0 \Rightarrow$$

$$\underbrace{\frac{1}{X} \frac{d^2 X}{dx^2}}_{\text{function of } x \text{ only}} + \underbrace{\frac{1}{Y} \frac{d^2 Y}{dy^2}}_{\text{function of } y \text{ only}} + \underbrace{\frac{1}{Z} \frac{d^2 Z}{dz^2}}_{\text{function of } z \text{ only}} = 0$$

The only possibility for the equality to hold is that each of

2

the three terms on the left-hand side is a constant:

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\alpha^2, \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = -\beta^2, \quad \frac{1}{Z} \frac{d^2 Z}{dz^2} = \alpha^2 + \beta^2$$

We then find three ordinary differential equations:

$$\begin{cases} \frac{d^2 X}{dx^2} + \alpha^2 X = 0 \\ \frac{d^2 Y}{dy^2} + \beta^2 Y = 0 \\ \frac{d^2 Z}{dz^2} - (\alpha^2 + \beta^2) Z = 0 \end{cases} \Rightarrow X \propto e^{\pm i\alpha x}, Y \propto e^{\pm i\beta y}, Z \propto e^{\pm \sqrt{\alpha^2 + \beta^2} z}$$

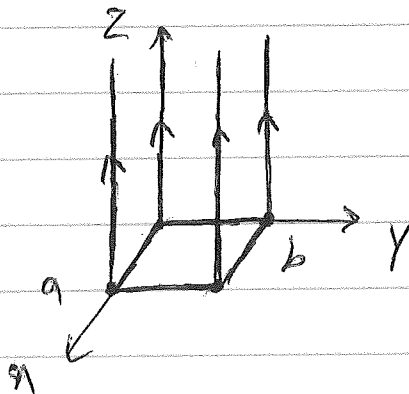
Note that α and β are completely arbitrary at this point and can be any complex number. Also, any linear combination of the products of X, Y, Z solutions will be a solution to the Laplace equation.

Example: A rectangular box with open top.

$$\Phi = 0 \text{ at } x=0, a$$

$$\Phi = 0 \text{ at } y=0, b$$

$$\Phi(x, y, z) = f(x, y)$$



(3)

For this problem, periodicity of the boundary condition with respect to x and y implies that α and β are real. Also, because $z = \infty$ is included, Z must be an exponentially decaying function of z .

Therefore:

$$\Phi(x, y, z) = \sum_{\alpha, \beta} (A_{\alpha} \sin \alpha x + B_{\alpha} \cos \alpha x) (C_{\beta} \sin \beta y + D_{\beta} \cos \beta y) e^{-\sqrt{\alpha^2 + \beta^2} z}$$

$$\Phi(x=0) = 0 \Rightarrow B_{\alpha} = 0$$

$$\Phi(y=0) = 0 \Rightarrow D_{\beta} = 0$$

$$\Phi(x=a) = 0 \Rightarrow \alpha_n = \frac{n\pi}{a} \quad n=1, 2, \dots$$

$$\Phi(y=b) = 0 \Rightarrow \beta_m = \frac{m\pi}{b} \quad m=1, 2, \dots$$

This results in:

$$\Phi(x, y, z) = \sum_{n, m=1}^{\infty} E_{nm} \sin\left(\frac{n\pi x}{a}\right) \left(\frac{m\pi y}{b}\right) e^{-\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} z}$$

Imposing the boundary condition on the x - y plane, we find:

$$f(x, y) = \sum_{n, m} E_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \Rightarrow$$

$$E_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) dx dy$$

In general, when more than one inhomogeneous boundary condition is given, the solution can be written as a superposition of solutions obtained for all but one boundary held at zero potential.

"Polar Coordinates"

$$\nabla^2 \Phi = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial \Phi}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0 \quad (\text{no } z \text{ dependence})$$

$$\Phi(s, \phi) = R(s) \psi(\phi) \Rightarrow \psi \frac{1}{s} \frac{d}{ds} \left(s \frac{dR}{ds} \right) + \frac{R}{s^2} \frac{d^2 \psi}{d\phi^2} = 0 \Rightarrow$$

$$\frac{s}{R} \frac{d}{ds} \left(s \frac{dR}{ds} \right) + \frac{1}{\psi} \frac{d^2 \psi}{d\phi^2} = 0$$

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$$\frac{1}{\psi} \frac{d^2 \psi}{d\phi^2} = -\nu^2$$

$$\frac{s}{R} \frac{d}{ds} \left(s \frac{dR}{ds} \right) = \nu^2$$

Hence:

$$\frac{d^2 \psi}{d\phi^2} + \nu^2 \psi = 0 \Rightarrow \psi = e^{\pm i\nu \phi}$$

$$s \frac{d}{ds} \left(s \frac{dR}{ds} \right) - \nu^2 R = 0 \Rightarrow s^2 \frac{d^2 R}{ds^2} + s \frac{dR}{ds} - \nu^2 R = 0$$

Taking $R \propto s^\lambda$, we find:

$$\lambda(\lambda-1) s^\lambda + \lambda s^\lambda - \nu^2 s^\lambda = 0 \Rightarrow \lambda = \pm \nu$$

Thus, the general solution is:

$$\Phi(s, \phi) = \sum_{\nu \neq 0} (A_\nu s^\nu + B_\nu s^{-\nu}) (C_\nu e^{i\nu\phi} + D_\nu e^{-i\nu\phi}) + (A + B \ln s)$$

We note that for $\nu=0$, the differential equation for R has a solution given as $A + B \ln s$. The only solution of Ψ that is physically acceptable for $\nu=0$ is a constant.

In problems where the full range $\phi = (0, 2\pi)$ is involved, Φ must be periodic with respect to ϕ . This implies that:

$$e^{\pm i\nu 2\pi} = 1 \Rightarrow \nu = m \quad m: \text{a non-zero integer}$$

In this case, therefore, we have:

$$\Phi(s, \phi) = (A + B \ln s) + \sum_{m \neq 0} (A_m s^m + B_m s^{-m}) e^{im\phi}$$

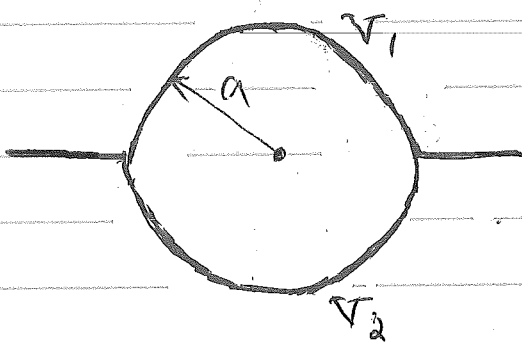
(6)

For problems where inside of a volume is of interest, ∇^2 powers of ρ are not permitted. This excludes ρ^{-m} terms for $m > 0$ and ρ^m terms for $m < 0$. Similarly, $\ln \rho$ will not be permitted in such cases. The solution then has the following form:

$$\bar{\Phi}(\rho, \phi) = \sum_{m=-\infty}^{+\infty} A_m \rho^{|m|} e^{im\phi}$$

Example: An infinite hollow cylinder with potential V_1 on one half and potential V_2 on the other half.

$$\bar{\Phi} = \begin{cases} V_1 & \rho = a, 0 < \phi < \pi \\ V_2 & \rho = a, \pi < \phi < 2\pi \end{cases}$$



We start from the following:

$$\bar{\Phi}(\rho, \phi) = \sum_{m=-\infty}^{+\infty} A_m \rho^{|m|} e^{im\phi}$$

From the boundary condition, we find:

$$A_m a^{|m|} = \frac{1}{2\pi} \left[\int_0^{\pi} V_1 e^{-im\phi} d\phi + \int_{\pi}^{2\pi} V_2 e^{-im\phi} d\phi \right] \Rightarrow$$

$$A_m = \frac{1}{a^{|m|}} \left[\int_0^{\pi} V_1 e^{-im\phi} d\phi + \int_{\pi}^{2\pi} V_2 e^{-im\phi} d\phi \right]$$

In the special case that $V_1 = V_2 = V$ (i.e., constant potential on the entire cylinder), we find:

$$A_m = V \delta_{m,0}$$

This implies that $\Phi = V$ everywhere inside the cylinder. The same situation holds for the exterior problem when outside the cylinder is volume of interest.